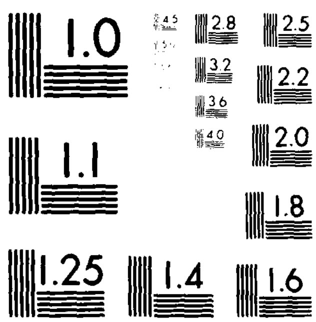
AD-A085 993 MASSACHUSETTS UNIV AMHERST LAB OF PSYCHOMETRIC AND --ETC F/G 12/1
BAYESIAN ESTIMATION IN THE ONE-PARAMETER LATENT TRAIT MODEL.(U)
MAR 80 H SWAMINATHAN, J A GIFFORD N00014-79-C-0039
UNCLASSIFIED LR-106 NL

1 OF 1
AD
A085993

END
DATE
FILMED
8-80
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

ADA085993

BAYESIAN ESTIMATION IN THE ONE-PARAMETER LATENT TRAIT MODEL

LEVEL II

HARIHARAN SWAMINATHAN

AND

JANICE A. GIFFORD

Research Report 80-1
March 1980

Laboratory of Psychometric and
Evaluative Research (LR 106)
School of Education
University of Massachusetts
Amherst, MA 01003

Prepared under contract No. N00014-79-C-0039, NR150-427
with the personnel and Training Research Programs
Psychological Sciences Division
Office of Naval Research

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report 80-1	2. GOVT ACCESSION NO. AD-A085993	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) 6 BAYESIAN ESTIMATION IN THE ONE-PARAMETER LATENT TRAIT MODEL.		5. TYPE OF REPORT & PERIOD COVERED 9 Research ~~Technical~~ Report
		6. PERFORMING ORG. REPORT NUMBER Lab. Report 106
7. AUTHOR(s) 10 Hariharan Swaminathan & Janice A. Gifford		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-79-C-0039 new
9. PERFORMING ORGANIZATION NAME AND ADDRESS Laboratory of Psychometric and Evaluative Research ~~School of Education~~/University of Massachusetts Amherst, MA 01003		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR150-427
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 458) Arlington, VA 22217	11	12. REPORT DATE March ~~1980~~
		13. NUMBER OF PAGES 36
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 12 52		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

14 LR-106, RR-80-1

18. SUPPLEMENTARY NOTES

This research was jointly supported by the Air Force Human Resources Laboratory and by the Office of Naval Research.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

latent trait theory
Bayesain estimation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

When several parameters are to be estimated simultaneously, and when both structural and incidental parameters have to be estimated, a Bayesian solution to the estimation problem may be appropriate. This is the case in latent trait models, where the "structural" parameters are item parameters, while the "incidental parameters" are ability parameters since these increase without bound as the numbers of examinees is increased to provide stable estimates of the item parameters. Bayesian estimates for the parameters in the one-parameter

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

411 294

latent trait model were obtained for two cases:

(1) Conditional estimation of ability (for those situations when items are previously calibrated), and

(2) Joint estimation of item and ability parameters.

For each of the two cases, a simulation study was carried out to study the efficacy of the two Bayesian procedures described and to compare the Bayesian estimates with the comparable maximum likelihood estimates. The Bayesian and maximum likelihood estimates were compared with respect to: (a) the mean value of the estimates, as compared with the mean values of the true values; (b) the mean squared error difference between true values and estimated values; and (c) the regression of the true value on the estimated value. Overall, the results favored the Bayesian estimates; the means of the estimates are closer to the means of the true values; the slopes and intercepts are in general closer to one and zero respectively; and the mean square deviations are dramatically smaller (in some cases, one-tenth the size of those for ML estimates).

TABLE OF CONTENTS

Bayesian Estimation in the One-Parameter Latent Trait Model[1,2]

INTRODUCTION

In recent years there has been considerable interest among measurement theorists and practitioners in latent trait theory since it offers the potential for improving educational and psychological measurement practices. However, before latent trait theory can be successfully applied to solve existing measurement problems, the problem of estimating parameters in latent trait models has to be addressed.

The literature in latent trait theory abounds with procedures for the estimation of parameters. The estimation procedures that have been developed over the past thirty years range from heuristic procedures such as those given by Urry (1974) and Jensema (1976) to conditional as well as unconditional maximum likelihood procedures (Andersen, 1970, 1972, 1973a, 1973b; Bock, 1972; Lord, 1968, 1974; Samejima, 1969, 1972; Wright & Panchapakesan, 1969; Wright & Douglas, 1977). With the exception of the "conditional" maximum likelihood procedure provided by Andersen (1970) for the one-parameter model, the maximum likelihood estimators of the parameters in the latent trait models are less than optional as a result

---

[1]The research reported here was performed pursuant to Grant No. N0014-79-C-0039 with the Office of Naval Research and to Grant No. FQ 7624-79-0014 with the Air Force Human Resources Laboratory. The opinions expressed here, however, do not reflect the positions or policies of these agencies.

[2]The author is grateful to the encouragement and support provided by Dr. Malcolm Ree of the Air Force Human Resources Laboratory, and to Dr. Charles Davis of the Office of Naval Research.

of the problem of estimating "structural parameters" in the presence of "incidental parameters" (Andersen, 1970; Zellner, 1971, pp. 114-154). The "structural parameters" in latent trait models are the item parameters while the "incidental parameters" are the ability parameters since these increase without bound as the number of examinees is increased to provide stable estimates of the parameters. Furthermore, as Novick, Lewis, and Jackson (1973) have remarked, "in the estimation of many parameters some, by chance, can be expected to be substantially overestimated and the others substantially underestimated."

When several parameters have to be estimated simultaneously, and when, as in the present case, both structural and incidental parameters have to be estimated, a Bayesian solution to the estimation problem may be appropriate (Zellner, 1971, pp. 114-119). This is particularly true if prior information or belief about the parameters is available, since in this case, the incorporation of this information will certainly increase the "accuracy" or the meaningfulness of the estimates. An example of this was encountered by Lord (1968), where in order to prevent estimates of the item discrimination parameter from drifting out of bounds, it was necessary to impose limits on the range of values the parameter could take. Although the estimation procedure employed by Lord (1968) was not Bayesian, this illustrates the role of prior information in obtaining meaningful estimates. A further argument that can be advanced in favor of a Bayesian approach is that the logic of the Bayesian inferential procedure is more appealing than the classical, sampling theoretic, inferential procedure. As Zellner (1971, p. 362) has pointed out, "...there is no need to justify inference procedures in terms of their

behavior in repeated, *as yet unobserved,*[1] samples as is usually done in the sampling theory approach." Consequently, it is possible to make probabilistic statements about the parameters themselves, based on the information that is available.

Bayesian Procedures

It may be instructive to review briefly the Bayesian estimation procedure. Let $p(\underline{y}, \underline{\theta})$ denote the joint probability density function (pdf) for a random observation vector $\underline{y}$ and a parameter vector $\underline{\theta}$, also random. Then,

$$\begin{aligned} p(\underline{y}, \underline{\theta}) &= p(\underline{y}|\underline{\theta})\ p(\underline{\theta}) \\ &= p(\underline{\theta}|\underline{y})\ p(\underline{y}) \end{aligned}$$

where

$$p(\underline{\theta}|\underline{y}) = p(\underline{\theta})\ p(\underline{y}|\underline{\theta})/p(\underline{y})$$

or,

$$p(\underline{\theta}|\underline{y}) \propto p(\underline{\theta})\ p(\underline{y}|\underline{\theta}) \quad [1]$$

since $p(\underline{y}) \neq 0$ is a constant. Equation [1] is the essence of Bayes' Theorem and is of primary importance in the estimation of parameters and for drawing inferences concerning the parameters. The probability density function $p(\underline{\theta}|\underline{y})$ is the *posterior pdf* for the parameter vector $\underline{\theta}$, given the sample information or data, and $p(\underline{\theta})$ is the *prior pdf* for the vector $\underline{\theta}$. The quantity $p(\underline{y}|\underline{\theta})$ is a proper pdf as long as $\underline{y}$ is a random variable. However, the moment the vector $\underline{y}$ is realized, $p(\underline{y}|\underline{\theta})$ ceases

[1]The italics have been provided by the authors.

to have the interpretation as a pdf. In this case, $p(\underline{y}|\underline{\theta})$ is strictly a mathematical function of $\underline{\theta}$, well known as the *likelihood function*. Since the notation $p(\underline{y}|\underline{\theta})$ can be mistaken for a pdf, the likelihood function is often written as $L(\underline{y}|\underline{\theta})$, and sometimes, to emphasize the fact that it is a function of $\underline{\theta}$, as $L(\underline{\theta}|\underline{y})$. Thus, the expression given in [1] can be written as

$$[2] \qquad p(\underline{\theta}|\underline{y}) \propto L(\underline{\theta}|\underline{y})\; p(\underline{\theta}) \; .$$

It is interesting to note that if $p(\underline{\theta})$ is assumed to be a constant, i.e., the prior belief about $\underline{\theta}$ is summarized via a uniform distribution, the posterior pdf of $\underline{\theta}$ is proportional to the likelihood function. In a sense, this interpretation constitutes a Bayesian justification of maximum likelihood principle.

Once the prior belief about the parameter $\underline{\theta}$ is specified, the joint posterior pdf of the vector $\underline{\theta}$ given the data can be readily obtained. The posterior pdf of $\underline{\theta}$ contains all the information necessary for drawing inferences concerning $\underline{\theta}$ (jointly or individually) and for obtaining estimtes of $\underline{\theta}$ once a "loss function" is prescribed. For instance, if a squared-error loss function is deemed appropriate, then the mean of the posterior pdf of $\underline{\theta}$ can be taken as the estimator of $\underline{\theta}$. On the other hand, if a zero-one loss function is appropriate, then, the mode of the posterior pdf of $\underline{\theta}$ is the estimator of $\underline{\theta}$. Similarly, for the absolute deviation loss function, the median of the posterior pdf of $\underline{\theta}$ is the appropriate estimator.

The Bayesian procedure described above has been successfully applied in a variety of situations. For a sampling of these applications the reader is referred to Novick and Jackson (1974), and Zellner (1971).

However, Bayesian methods have found only a limited application in the area of latent trait theory. Birnbaum (1969) obtained Bayes estimates of the ability parameter in the one- and two-parameter logistic models under the assumption that the item parameters were known. He chose, for mathematical tractability, the prior pdf of θ_i, the ability of the ith examinee, to be the logistic density function, i.e.

$$p(\theta_i) = \exp(-D\theta_i)/[1+\exp(-D\theta_i)]^2$$

where D=1.7 is a scaling factor. Owen (1975), in applying the latent trait model in an adaptive testing context, obtained Bayes estimates of ability, θ_i, under the assumption that the prior pdf of θ_i was normal with mean, zero, and variance, unity.

The Bayesian procedure suggested by Birnbaum (1969) and Owen (1975) require rather exact specification of prior belief.[1] An alternative and a more powerful procedure has been suggested by Lindley (1971). He has shown that if the information that is available can be considered *exchangeable*, then a hierarchical Bayesian model can be effectively employed for the estimation of parameters.

In order to illustrate the hierarchical model, let us consider the problem of estimating, say, the ability θ_i of an individual ($i=1, \ldots, N$). If it can be assumed, apriori, that exchangeability holds, i.e., the information about θ_i is no different from the information about any other θ_j, observed or yet to be observed, then, θ_i can be assumed to be a random sample from some distribution, $p(\theta)$. For convenience, if $p(\theta)$ is taken

[1]Meredith and Kearnes (1973) and Sanathanan and Blumenthal (1978) have obtained empirical Bayes estimators of the ability parameters for the one-parameter model. In these procedures the prior pdf is estimated from the data.

to be normal with mean μ and variance σ^2, then this would constitute specification of the first stage of the hierarchical model. Since μ and σ^2 are unknown, specifying prior beliefs on these "hyperparameters" would constitute the second stage of the hierarchical model. Usually, the hyperparameter distributions are specified in such a way that they depend upon constants which can be determined from the prior belief the investigator has about the parameters, and hence the hierarchical model terminates at the second stage. With this two stage model, it is possible to estimate θ_i (i=1, ..., N) without any reference to the nuisance parameters, μ and σ^2.

Novick (1971) has described this hierarchical model as an analog of the empirical Bayes procedure advocated by Robbins (1955) and the simultaneous estimation procedure provided by Stein (1962). Furthermore, as Novick, Lewis, and Jackson (1973) have pointed out, this procedure not only employs the direct information gained through the observation of an individual, but also the collateral information contained in observations from other individuals. They further note that, "In effect, this collateral information is used to provide 'prior' information for the estimation.... Thus to some extent, the problem of selecting prior distributions for Bayesian analyses is neutralized, and this is effected from a strictly Bayesian approach."

The hierarchical Bayesian model has been successfully employed by Lindley and Smith (1972), Novick et al. (1973), and Zellner (1971), to name a few. However, this approach has not been employed for estimating parameters in latent trait models. The purpose of this paper, hence, is to provide a Bayesian estimation procedure, in the sense of Lindley, for estimating parameters in the one-parameter latent trait model.

Bayesian Estimation in the One-Parameter Logistic Model

The Model

Let X_{ij} denote a random variable that represents the binary response of an examinee i (i=1, ..., N) on item j (j=1, ..., n). If the examinee responds correctly to the item, $X_{ij}=1$, while for an incorrect response, $X_{ij}=0$. We assume that the complete latent space is unidimensional, and that the probability, $P[X_{ij}=1]$, that an individual with ability parameter θ_i will correctly respond to an item with difficulty parameter, b_j, is given by the logistic model,

$$[3] \qquad P[X_{ij}=1|\theta_i] = \exp(\theta_i-b_j)/\{1+\exp(\theta_i-b_j)\}.$$

On the other hand, the probability that the individual will respond incorrectly is given by

$$[4] \qquad \begin{aligned} P[X_{ij}=0|\theta_i] &= 1 - P[X_{ij}=1|\theta_i] \\ &= 1/\{1+\exp(\theta_i-b_j)\} . \end{aligned}$$

The probabilities given in Equations [3] and [4] can be combined to yield

$$[5] \qquad P[X_{ij} = x_{ij}|\theta_i] = \exp\{x_{ij}(\theta_i-b_j)\}/\{1+\exp(\theta_i-b_j)\}$$

where $x_{ij}=1$ for a correct response and $x_{ij}=0$ for an incorrect response.

The above model, since it depends only on one item parameter, difficulty, is commonly known as the Rasch model or the one-parameter logistic model. For a detailed description of this model and its properties, the reader is referred to Wright (1977).

Conditional Estimation of Ability

In some situations it may be of interest to estimate the ability θ_i of an examinee who takes a test which has been calibrated, i.e., the difficulty parameters are known. Moreover, since the problem of estimating ability when the item parameters are known is simpler to deal with and provides an illustration of the basic ideas involved, this case will be dealt with in detail first.

The model given by Equation [5], should in the strict sense be expressed as

$$[6] \qquad P[X_{ij} = x_{ij} | \theta_i, b_j] \;=\; \exp\{x_{ij}(\theta_i - b_j)\}/\{1+\exp(\theta_i - b_j)\} \; .$$

Although there are several ways to write the model, the expression given by [6] is the most convenient for the present situation.

It follows, from the principle of local independence, that the joint probability of responses of the N examinees on n items is given by

$$[7] \qquad P[X_{11}=x_{11}, X_{12}=x_{12}, \ldots, X_{ij}=x_{ij}, \ldots, X_{Nn}=x_{Nn} | \theta_1, \theta_2, \ldots, \theta_N; b_1, b_2, \ldots, b_n]$$
$$= \prod_{i=1}^{N} \prod_{j=1}^{n} \exp\{x_{ij}(\theta_i - b_j)\}/\{1+\exp(\theta_i - b_j)\} \; .$$

Once the responses of the N examinees on the n items are observed, the above expression ceases to have the probability interpretation and becomes the likelihood function, $L(\underline{X} = \underline{x} | \underline{\theta}, \underline{b})$. Upon simplification,

$$[8] \qquad L(\underline{X}=\underline{x}|\underline{\theta},\underline{b}) \;=\; \exp\{\sum_i \sum_j x_{ij}(\theta_i - b_j)\}/\Pi\;\Pi\{(1+\exp(\theta_i - b_j)\}$$
$$= \exp\{\sum_i r_i\theta_i - \sum_j q_j b_j\}/\Pi\;\Pi\{(1+\exp(\theta_i - b_j)\}$$

where $r_i = \sum_j x_{ij}$, and $q_j = \sum_i x_{ij}$. Since the item parameters are

known constants, the likelihood function is strictly a function of $\underline{\theta}$ and, hence, can be expressed as

$$[9] \qquad L(\underline{x}|\underline{\theta}, b) \propto \exp\{\sum_i r_i\theta_i\}/\Pi\ \Pi\{1+\exp(\theta_i-b_j)\}\ .$$

Returning to Equation [1], we see that in order to obtain the posterior density function of $\underline{\theta}$ given the observations and the item parameters, it is necessary to specify the prior distribution of $\underline{\theta}$. To this end, in the first stage of the hierarchical model, we assume that, apriori, the ability parameters, θ_i, are independently and identically normally distributed, i.e.,

$$[10] \qquad \theta_i|\mu,\phi \sim N(\mu,\phi).$$

The assumption that the thetas are independently and identically distributed follows from the assumption of exchangeable prior information about the thetas. The assumption of normality also appears to be reasonable and has been made by numerous authors, e.g., Lord and Novick (1968).

In order to complete the hierarchical Bayesian model, we have to specify prior distributions for μ and ϕ. This is the second stage. At this level, we assume that, apriori, μ and ϕ are independently distributed, and that μ has the uniform distribution. Thus,

$$[11] \qquad p(\mu,\phi) \propto p(\phi).$$

The uniform distribution is not a proper distribution, although this choice can be justified to some extent (Zellner, 1971, pp. 41-43). It may, however, be more appropriate to specify a "non-diffuse" prior and this possibility will be explored further in a later paper.

It now remains to specify the form of $p(\phi)$. Since ϕ is the variance of θ_i, ϕ can be assumed to have the inverse chi-square, χ^{-2} distribution, i.e.,

$$[12] \qquad p(\phi|\nu, s^2) \propto \phi^{-(\frac{\nu}{2}+1)} \exp(-\nu s^2/2\phi).$$

The quantities ν and s^2 are parameters of the inverse chi-square distribution, and have to be specified apriori. The inverse chi-square distribution can be expressed in different ways. Novick and Jackson (1974) prefer the form

$$p(\phi|\nu, \lambda) \propto \phi^{-(\frac{\nu}{2}+1)} \exp(-\lambda/2\phi).$$

For this form, the mean of the distribution is $\lambda/(\nu-2)$ and the mode is $\lambda/(\nu+2)$. For the form given by Equation [12] the mean is $s^2\nu/(\nu-2)$ and the mode is $s^2\nu/(\nu+2)$, with both mean and mode approaching s^2 as ν increases. These two forms are clearly equivalent, but the form given by Equation [12] is employed in the sequel because it provides a direct interpretation of the parameter ν and s^2. The quantity s^2 thus represents the investigator's belief about the "typical" value of the parameter ϕ while ν represents his/her degree of confidence.

The joint posterior distribution of $\underline{\theta}' = [\theta_1, \theta_2, \ldots, \theta_N]$ given $\underline{b}$ and the item responses is given by

$$[13] \qquad p(\underline{\theta}|\underline{b},\underline{x}) \propto L(\underline{x}|\underline{\theta},\underline{b})\; p(\underline{\theta}|\mu,\phi)\rho(\mu,\phi).$$

The likelihood function $L(\underline{x}|\underline{\theta},\underline{b})$ is given by Equation [9], $p(\mu,\phi)$ by Equation [12], and

$$[14] \qquad p(\underline{\theta}|\mu,\phi) = \prod_{i=1}^{N} \phi^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\theta_i-\mu)^2/\phi\}$$

$$= \phi^{-\frac{N}{2}} \exp\{-\sum_{i=1}^{N}(\theta_i-\mu)^2/2\phi\}$$

Combining these expressions, we have,

$$[15] \qquad p(\underline{\theta}|\underline{b},\underline{x},\mu,\phi,\nu, s^2) \propto [\exp\{\sum_i r_i\theta_i\}/\prod_i\prod_j\{1+\exp(\theta_i-b_j)\}]$$

$$[\phi^{-\frac{N}{2}} \exp\{-\sum_i(\theta_i-\mu)^2/2\phi\}][\phi^{-(\frac{\nu}{2}+1)} \exp(-\nu s^2/2\phi)]$$

The above expression depends upon the "nuisance" parameters μ and ϕ and hence these have to be integrated out. Since $\sum(\theta_i-\mu)^2 = \sum(\theta_i-\theta_.)^2 + N(\theta_.-\mu)^2$, and

$$\int_{-\infty}^{\infty} \exp -\{N(\theta.-\mu)^2/2\phi\}d_\mu \propto \phi^{\frac{1}{2}},$$

integration with respect to μ yields

$$[16] \qquad p(\underline{\theta}|\underline{b},\underline{x},\phi,\nu,s^2) \propto L(\underline{x}|\underline{\theta},\underline{b})\ \phi^{-(N+\nu+1)/2} \exp[-\{\nu s^2+\sum(\theta_i-\theta.)^2\}/2\phi] .$$

Noting that

$$\int_{0}^{\infty} \phi^{-m} \exp(-k/\phi)d\phi \propto k^{-(m-1)}$$

and integrating with respect to ϕ, we obtain

$$[17] \qquad p(\underline{\theta}|\underline{b},\underline{x},\nu,s^2) \propto L(\underline{x}|\underline{\theta},\underline{b}) \quad \{\nu s^2 + \sum_i(\theta_i-\theta_.)^2\}^{-(N+\nu-1)/2}$$

$$[18] \qquad = [\exp\{\sum_i r_i\theta_i\}/\prod_i\prod_j\{1+\exp(\theta_i-b_j)\}]$$

$$\cdot\{\nu s^2 + \sum_i(\theta_i-\theta_.)^2\}^{-(N+\nu-1)/2}$$

The joint posterior modes are obtained by differentiating log $p(\theta|\underline{b},\underline{x})$ with respect to $\underline{\theta}$, setting these derivatives equal to zero, and solving the resulting equations:

$$[19] \qquad \sum_{j=1}^{n} P_{ij} = r_i - (\theta_i - \theta_.)/\sigma^2 \qquad (i=1, \ldots, N)$$

where

$$P_{ij} = \exp(\theta_i - b_j)/\{1+\exp(\theta_i - b_j)\}$$

and

$$\sigma^2 = \{\nu s^2 + \sum_{i=1}^{N} (\theta_i - \theta_.)^2\}/(\nu+N-1) \quad .$$

Since this system of equations is non-linear, numerical procedures have to be employed. The Newton-Raphson iterative procedure is ideally suited for this situation. Let

$$[20] \qquad f(\theta_i) = \sum_{j=1}^{n} P_{ij} + (\theta_i - \theta_.)/\sigma^2 - r_i \quad .$$

Then

$$[21] \qquad f'(\theta_i) = \sum_{j=1}^{n} P_{ij}(1-P_{ij}) + \{\sigma^2(1-\tfrac{1}{N}) - 2(\theta_i - \theta_.)/(\nu+N-1)\}/(\sigma^2)^2.$$

If $\theta_i^{(k)}$ is the value of θ_i at the kth iteration, then $\theta_i^{(k+1)}$ is given by

$$[22] \qquad \theta_i^{(k+1)} = \theta_i^{(k)} - f(\theta_i^{(k)})/f'(\theta_i^{(k)}),$$

with $\theta_i^{(o)}$, the starting value being given by (Wright & Douglas, 1977),

$$[23] \qquad \theta_i^{(o)} = b_. + \{1+s_b^2/2.89\}\log(r_i/n-r_i)$$

where

$$b_. = \sum b_j/n, \text{ and } s_b^2 = \sum(b_j - b_.)^2/(n-1) \quad .$$

Although the iterative scheme given in [22] is for estimating the ability θ_i for each individual, in reality, only the ability corresponding to each raw score r (r=1, ..., n-1) need be estimated. The ability corresponding to raw score r=0 and r=n cannot be estimated by virtue of [23]. Hence, individuals who obtain perfect score or zero score are eliminated from the analysis. It should also be pointed out the Newton-Raphson scheme given above is not the vector version of the procedure since for this procedure the matrix of derivatives $\{\partial f/\partial\theta_i\partial\theta_j\}$ has to be computed and inverted. The procedure described here worked sufficiently well, converging in as few as three to four iterations.

Joint Estimation of Item and Ability Parameters

The case considered above, where the item parameters were assumed to be known, provides the necessary background for the Bayesian estimation procedure. However, this situation may not be realistic and, hence, it is necessary to develop a procedure for the joint estimation of the item and ability parameters.

We proceed in the manner indicated for the case of known item parameters. Hence, in addition to making the assumptions about the ability parameters, we have to make assumptions regarding the item parameters. Again, as in the previous case, we specify prior beliefs about the parameters in two stages. In the first stage, for the model given in [5], we assume:

[24a] $\theta_i|\mu_\theta,\phi_\theta \sim N(\mu_\theta,\phi_\theta)$, (i=1, ..., N)

[24b] $b_j|\mu_b,\phi_b \sim N(\mu_b,\phi_b)$. (j=1, ..., n)

In addition, we assume that, apriori, θ_i and b_j are independent, θ_k and θ_ℓ $(k\neq\ell)$ are independent, and b_k and b_ℓ are independent.

As for the ability parameters, the specification of prior belief about b_j seems reasonable, especially if an item bank is available. This assumption has been made by several authors (Lord & Novick, 1968; Wright & Douglas, 1977). Furthermore, as a result of the hierarchical Bayesian model, departures from this assumption appear to have a negligible effect on the estimates of b_j.

For the second stage, we assume that

$$[25a] \quad p(\mu_\theta,\phi_\theta) \propto p(\phi_\theta) \propto \phi_\theta^{-(\nu_\theta/2+1)} \exp(-s_\theta^2\nu_\theta/2\phi_\theta),$$

and

$$[25b] \quad p(\mu_b,\phi_b) \propto p(\phi_b) \propto \phi_b^{-(\nu_b/2+1)} \exp(-s_b^2\nu_b/2\phi_b).$$

We have thus assumed that, apriori, the hyperparameters are independent, and that the prior information about the parameters, μ_θ and μ_b, is "vague".

The joint posterior pdf of $\underline{\theta}$, and $\underline{b}$, is given by

$$[26] \quad p(\underline{\theta},\underline{b}|\underline{x},\mu_\theta,\phi_\theta,\mu_b,\phi_b,\nu_\theta,s_\theta^2,\nu_b,s_b^2) \propto L(\underline{\theta},\underline{b}|\underline{x})\{\prod_{i=1}^{N} p(\theta_i) \prod_{j=1}^{n} p(b_j)\}p(\phi_\theta)\ p(\phi_b)$$

where $L(\underline{\theta},\underline{b}|\underline{x})$ is the likelihood function given by [8]. Now

$$[27] \quad \{\prod_{i=1}^{N} p(\theta_i)\}\ p(\phi_\theta) \propto \phi^{-(N+\nu_\theta+2)/2} \exp(-\nu_\theta s_\theta^2/2\phi_\theta)\ \exp\{-(\theta_i-\mu_\theta)^2/2\phi_\theta\}\ .$$

Upon integrating with respect to ϕ_θ and μ_θ, we have, from [17]

[28] $$\int_{-\infty}^{\infty} \int_{0}^{\infty} \{\prod_{i=1}^{N} p(\theta_i)\} \, p(\phi_\theta) d\mu_\theta \, d\phi_\theta \propto [\nu_\theta s_\theta^2 + \sum_{i=1}^{N} (\theta_i - \theta_.)^2]^{-(N+\nu_\theta-1)/2} .$$

Similarly,

[29] $$\int_{-\infty}^{\infty} \int_{0}^{\infty} \{\prod_{j=1}^{n} p(b_j)\} \, p(\phi_b) d\mu_b \, d\phi_b \propto [\nu_b s_b^2 + \sum_{j=1}^{n} (b_j - b_.)^2]^{-(n+\nu_b-1)/2}$$

Combining [26], [28] and [29], we obtain the joint posterior density of $\underline{\theta}$ and $\underline{b}$:

[30] $$p(\underline{\theta},\underline{b}|\underline{x},\nu_\theta,s_\theta^2,\nu_b,s_b^2)$$
$$\propto [\{\exp(\sum_{i=1}^{N} r_i\theta_i)\}\{\nu_\theta s_\theta^2 + \sum_{i=1}^{N}(\theta_i-\theta_.)^2\}^{-(N+\nu_\theta-1)/2}]$$
$$\cdot [\{\exp(-\sum_{j=1}^{n} q_j b_j)\}\{\nu_b s_b^2 + \sum_{j=1}^{n} (b_j-b_.)\}^{-(n+\nu_b-1)/2}]$$
$$\cdot [\prod_{i=1}^{N}\prod_{j=1}^{n} \{1 + \exp(\theta_i-b_j)\}]^{-1} .$$

The quantity given as $L(\underline{\theta},\underline{b}|\underline{x})$,

$$\exp(\sum r_i\theta_i)\exp(\sum q_j b_j)/\Pi\,\Pi\{1+\exp(\theta_i-b_j)\} = \Pi\,\Pi\exp(\theta_i-b_j)/\{1+\exp(\theta_i-b_j)\},$$

and, hence, is bounded. In fact,

$$|L(\underline{\theta},\underline{b}|\underline{x})| \leq 1 .$$

Therefore, it follows that

$$\int |p(\underline{\theta},\underline{b}|\ldots)|\, d\underline{\theta} d\underline{b} < \int [\nu_\theta s_\theta^2 + \sum_{i=1}^{N} (\theta_i - \theta_.)^2]^{-(N+\nu_\theta-1)/2} d\underline{\theta}$$

$$\cdot \int [\nu_b s_b^2 + \sum_{j=1}^{n} (b_j - b_.)^2]^{-(n+\nu_b-1)/2} d\underline{b} \quad .$$

The integrals on the right of the inequality clearly exist since the kernels are those of multivariate t densities. Hence, the posterior pdf, $p(\underline{\theta},\underline{b}|\underline{x},\nu_\theta,s_\theta^2,\nu_b,s_b^2)$, is a proper pdf although the normalizing constant cannot be evaluated explicitly.

The joint posterior modes may be taken as estimates of θ_i and b_j ($i=1, \ldots, N$; $j=1, \ldots, n$). These are obtained by setting equal to zero the derivatives of $\log p(\underline{\theta},\underline{b}|\ldots)$, and solving the resulting equations:

$$[31] \qquad \sum_{j=1}^{n} P_{ij} = r_i - (\theta_i - \theta_.)/\sigma_\theta^2 \qquad (i=1, \ldots, N),$$

$$[32] \qquad \sum_{i=1}^{N} P_{ij} = q_j + (b_j - b_.)/\sigma_b^2 \qquad (j=1, \ldots, n),$$

where

$$P_{ij} = \exp(\theta_i - b_j)/\{1 + \exp(\theta_i - b_j)\} \quad ,$$

$$r_i = \sum_j x_{ij} \quad ,$$

$$q_j = \sum_i x_{ij} \quad ,$$

$$\sigma_\theta^2 = \{\nu_\theta s_\theta^2 + \sum_i (\theta_i - \theta_.)^2\}/(\nu_\theta + N - 1) \quad ,$$

and

$$\sigma_b^2 = \{\nu_b s_b^2 + \sum_j (b_j - b_.)\}^2/(\nu_b + n - 1) \quad .$$

Since the systems of equations is non-linear, the Newton-Raphson procedure is employed to solve the equations iteratively. In order to accomplish this, we let

$$[33] \qquad f(\theta_i) = \sum_{j=1}^{n} P_{ij} + (\theta_i - \theta_.)/\sigma_\theta^2 - r_i$$

and

$$[34] \qquad h(b_j) = \sum_{i=1}^{N} P_{ij} - (b_j - b_.)/\sigma_b^2 - q_j \quad .$$

Then

$$[35] \qquad f'(\theta_i) = \sum_{j=1}^{n} P_{ij}(1-P_{ij}) + \{\sigma_\theta^2(1 - \tfrac{1}{N}) - 2(\theta_i - \theta_.)/(\nu_\theta + N-1)\}/(\sigma_\theta^2)^2 \ ,$$

and

$$[36] \qquad h'(b_j) = -\sum_{i=1}^{N} P_{ij}(1-P_{ij}) - \{\sigma_b^2(1 - \tfrac{1}{n}) - 2(b_j - b_.)/(\nu_b + n-1)\}/(\sigma_b)^2 \ .$$

As before, if $\theta_i^{(k)}$ and $b_j^{(k)}$ denote the values of θ_i and b_j at the kth iteration, then

$$[37] \qquad \theta_i^{(k+1)} = \theta_i^{(k)} - f(\theta_i^{(k)})/f'(\theta_i^{(k)}) \quad ,$$

and

$$[38] \qquad b_j^{(k+1)} = b_j^{(k)} - h(b_j^{(k)})/h'(b_j^{(k)}).$$

Starting with initial values $\theta_i^{(o)}$ $(i=1, \ldots, N)$, and $b_j^{(o)}$ $(j=1, \ldots, n)$, where $\theta_i^{(o)}$ is given by [23], and

$$b_j^{(o)} = \log [(N-q_i)/q_i]$$

$\underline{\theta}$ is estimated. These values of $\underline{\theta}$ are then used to obtain revised estimates of $\underline{b}$. This process is repeated with the revised estimates of $\underline{b}$ being used to obtain revised estimates of $\underline{\theta}$. The process is terminated when the convergence criterion is reached. This procedure is not the full Newton-Raphson procedure and, in this case, is preferred to the full Newton-Raphson procedure since the latter requires obtaining an inverse of the

matrix of second derivatives at each stage of the iteration. In practice, the procedure outlined here converges rather rapidly.

As pointed out earlier, although the equations provided are for estimating $\theta_i (i=1, \ldots, N)$, only $\theta_r (r=1, \ldots, n-1)$ need be estimated. In order to carry this out, the quantities given in Equations [34] and [35] have to be computed as follows:

$$\sum_{i=1}^{N} P_{ij} \simeq \sum_{r=1}^{n-1} N_r P_j(\theta_r)$$

$$\sum_{i=1}^{N} P_{ij}(1-P_{ij}) \simeq \sum_{r=1}^{n-1} N_r P_j(\theta_r)\{1-P_j(\theta_r)\}$$

where N_r denotes the number of examinees who obtained raw score r and

$$P_j(\theta_r) = \exp(\theta_r - b_j)/\ \{1 + \exp(\theta_r - b_j)\} .$$

Large Sample Properties of the Posterior Distribution

The posterior pdf, $p(\underline{\theta},\underline{b}|\nu_\theta,\nu_b,s_\theta^2,s_b^2,\underline{x})$, given by Equation [30] is a product of the likelihood function and a multivariate "double-t" distribution. The "double-t" distribution is a product of two multivariate t densities (Tiao & Zellner, 1964; Zellner, 1971, p. 101). As a result of its complex form, properties of the posterior pdf cannot be obtained. However, it is possible to obtain the asymptotic properties of the posterior pdf, and this will suffice, in most cases, for inferences to be drawn regarding the parameters.

Let $\underline{t}$ be a vector of parameters, and $\underline{y}$ a vector of observations. Then, the posterior pdf of $\underline{t}$, $p(\underline{t}|\underline{y})$, is

$$p(\underline{t}|\underline{y}) \propto p(\underline{t}) \quad L(\underline{y}|\underline{t})$$

where p(t) is the prior distribution of $\underline{t}$ and $L(\underline{y}|\underline{t})$, the likelihood function. Then, for large samples,

$$[39] \qquad p(\underline{t}|\underline{y}) \propto L(\underline{y}|\underline{t}) \ ,$$

and, in turn, $L(\underline{y}|\underline{t})$ is approximately multivariate normal centered at $\hat{\underline{t}}$, the maximum likelihood estimate, with dispersion matrix

$$[40] \qquad \Sigma = [-\partial^2 \log L(\underline{y}|\underline{t})/\partial t_i \partial t_j]^{-1}_{\underline{t}=\hat{\underline{t}}}$$

Thus, for large samples,

$$\underline{t}|\underline{y} \sim N(\hat{\underline{t}}, \Sigma_{\underline{t}=\hat{t}}) \ .$$

For a detailed discussion of this result we refer the reader to Jeffreys (1961, p. 193) and Zellner (1971, p. 32).

This result clearly applies in the present situation when both n and N, the number of items and the number of examinees, are large. Denoting the [(n+N)x1] vector $[\underline{\theta},\underline{b}]$ as

$$\underline{t}' = [\underline{\theta}' \ \underline{b}'] \ ,$$

$$[41] \qquad \underline{t}|\underline{x} \sim N(\hat{\underline{t}}, \Sigma) \ .$$

In order to evaluate Σ, we write

$$[42] \qquad \Sigma = \begin{bmatrix} G_\theta & G_{\theta b} \\ G_{b\theta} & G_b \end{bmatrix}^{-1}$$

where

$$[43] \quad G_\theta = \{-\partial^2 \log L(\underline{x}|\underline{\theta},\underline{b})/\partial\theta_\ell\partial\theta_m\}_{\underline{\theta}=\hat{\underline{\theta}}}$$

$$= \{\sum_{j=1}^{n} P_{\ell j}(1-P_{mj})\}\ \delta_{\ell m}$$

where $\delta_{\ell m}$ is the Kronecker delta,

$$[44] \quad G_b = \{-\partial^2 \log L(\underline{x}|\underline{\theta},\underline{b})/\partial b_\ell \partial b_m\}_{\underline{b}=\hat{\underline{b}}}$$

$$= \{\sum_{i=1}^{N} P_{i\ell}(1-P_{im})\}\ \delta_{\ell m},$$

and

$$[45] \quad G_{\theta b} = \{-\partial^2 \log L(\underline{x}|\underline{\theta},\underline{b})/\partial\theta_i \partial b_j\}_{\underline{\theta}=\hat{\underline{\theta}},\underline{b}=\hat{\underline{b}}}$$

$$= P_{ij}(1-P_{ij}) \quad .$$

Thus, the marginal distribution of θ_i has mean $\hat{\theta}_i$, the maximum likelihood estimate of θ_i, and variance, $\sigma^2_{\theta_i}$, given by the ith diagonal element of Σ, i.e.,

$$[46] \quad \sigma^2_\theta = [G_\theta - G_{\theta b}\ G_b^{-1}\ G_{b\theta}]^{-1}_{ii} \quad .$$

Similarly, the marginal distribution of b_j has mean $\hat{b}_j$, the maximum likelihood estimate of b_j, and variance, $\sigma^2_{b_j}$, given the jth diagnonal element of Σ, i.e.,

$$[47] \quad \sigma^2_{b_j} = [G_b - G_{b\theta}\ G_\theta^{-1}\ G_{\theta b}]^{-1}_{jj} \quad .$$

This approximation to the posterior pdf of $\underline{\theta}$ and $\underline{b}$ can be improved upon if we take into account the "double-t" distribution (see Equation [30]). For a sufficiently large sample, the multivariate t density approaches the normal density. Thus, in the expression

$$[\nu_\theta s_\theta^2 + \sum_{i=1}^{N} (\theta_i - \theta_.)^2]^{-(\nu_\theta+N-1)/2}$$

if we write $\nu_\theta = Nk_\theta$ where $0<k_\theta\leq 1$, for large N, we obtain

$$[48] \qquad [Nk_\theta s_\theta^2 + \sum_{i=1}^{N} (\theta_i - \theta_.)^2]^{-N(k_\theta+1)/2+\frac{1}{2}} \simeq \exp[-\frac{(k_\theta+1)}{2k_\theta s_\theta^2} \sum_{i=1}^{N} (\theta_i - \theta_.)^2]$$

$$= \exp\{-\frac{1}{2}\,\underline{\theta}'\,A_{11}\,\underline{\theta}\}$$

where

$$A_{11} = \frac{(k_\theta+1)}{k_\theta s_\theta^2}\,[I_N - \frac{1}{N}\,\underline{1}\,\underline{1}']$$

with I_N being the identity matrix and $\underline{1}' = [1\ 1\ 1\ \ldots\ 1]$. Similarly, for large n,

$$[49] \qquad [\nu_b s_b^2 + \sum_{j=1}^{n} (b_j - b_.)^2]^{-(\nu_b+n-1)/2} = [nk_b s_b^2 + \sum_{j=1}^{n} (b_j - b_.)^2]^{-n(k_b+1)/2+\frac{1}{2}}$$

$$\simeq \exp\{-\frac{(k_b+1)}{2k_b s_b^2} \sum_{j=1}^{n} (b_j - b_.)^2\}$$

$$= \exp\{-\frac{1}{2}\,\underline{b}\,A_{22}\,\underline{b}\}$$

where

$$A_{22} = \frac{(k_b+1)}{k_b s_b^2}\,[I_n - \frac{1}{n}\,\underline{1}\,\underline{1}'] \quad .$$

Thus,

$$[50] \quad [\nu_b s_b^2 + \sum_{j=1}^{n} (b_j - b_.)^2]^{-(\nu_b+n-1)/2} [\nu_\theta s_\theta^2 + \sum_{i=1}^{N} (\theta_i - \theta_.)^2]^{-(\nu_\theta+N-1)/2}$$

$$\propto \exp \{-\frac{1}{2} (\underline{\theta}' A_{11} \underline{\theta} + \underline{b}' A_{22} \underline{b})$$

$$[51] \quad = \exp \{-\frac{1}{2} \underline{t}' A \underline{t}\}$$

where

$$\underline{t}' = [\ \underline{\theta}'\ \underline{b}'\ ] \ ,$$

and

$$A = \begin{vmatrix} A_{11} & 0 \\ 0 & A_{22} \end{vmatrix}$$

Combining [41] and [51], we have

$$[52] \quad p(\underline{\theta}, \underline{b} | \underline{x}, \nu_\theta, \nu_b, s_\theta^2, s_b^2)$$

$$[53] \quad \propto \exp -\frac{1}{2} \{(\underline{t}-\underline{\hat{t}})' \Sigma (\underline{t}-\underline{\hat{t}}) + \underline{t}' A \underline{t}\}$$

$$[54] \quad = \exp\{-\frac{1}{2} (\underline{t} - \underline{\tau})' T (\underline{t} - \underline{\tau})\}$$

where

$$[55] \quad T = \Sigma + A$$

and

$$[56] \quad \underline{\tau} = (\Sigma + A)^{-1} (\Sigma \underline{t})^*$$

---

*This result follows from the fact that

$$(\underline{x}-\underline{a})' A (\underline{x}-\underline{a}) + (\underline{x}-\underline{b})' B (\underline{x}-\underline{b}) = (\underline{x}-\underline{t})' T (\underline{x}-\underline{t}) + \text{constant},$$

where

$$T = A+B,$$

and

$$\underline{t} = (A+B)^{-1} \{A\underline{a} + B\underline{b}\} \ .$$

If the off diagonal matrix $G_{\theta b}$ in [42] can be ignored, then

$$[57] \qquad \sigma^2_{\theta_i} = [\sum_{j=1}^{n} P_{ij}(1-P_{ij})]^{-1} + (N-1)(k_\theta+1)/Nk_\theta s^2_\theta$$

and

$$[58] \qquad \sigma^2_{b_j} = [\sum_{i=1}^{N} P_{ij}(1-P_{ij})]^{-1} + (n-1)(k_b+1)/nk_b s^2_b \quad .$$

The expression [57] is useful when the item parameters are considered known. Similarly, [58] is applicable when the ability parameters are known. In general, however, when the item and ability parameters are estimated simultaneously, the off diagonal matrix, $G_{\theta b}$, cannot be ignored, and hence, in this case, the complete expression given by either [46] or [55] should be employed. With these results it is possible to construct "credibility intervals" (Novick & Jackson, 1974) for the parameters of interest.

COMPARISON STUDIES

In order to study the efficacy of the Bayesian procedure described above and to compare the Bayesian estimates with the maximum likelihood estimates, a simulation study was carried out. Although simulation studies may not be realistic in some situations, they can be justified in the present context since only through a simulation study can one estimation procedure be compared with another.

Artificial data, representing the responses of N individuals on n items, were generated using DATGEN (Hambleton & Rovinelli, 1973) according to the one-parameter logistic model. In generating the values of

θ_i and b_j ($i=1, \ldots, N$; $j=1, \ldots, n$), it was assumed that θ_i and b_j were independently and identically normally distributed with mean, zero, and variance, unity (we shall return to a discussion of this issue later). The design of the comparison study was conceptualized in terms of the following, completely crossed, factors: estimation procedure (Bayesian, maximum likelihood); number of examinees, N (20, 50); number of items, n (15, 25, 40, 50). This design was carried out for (i) conditional estimation of $\underline{\theta}$, and, (ii) joint estimation of $\underline{\theta}$ and $\underline{b}$.

The size of the examinee population, N, and the test length, n, were chosen to facilitate comparison of the maximum likelihood and the Bayesian estimates for small sample sizes and short tests, since the large sample behavior of the maximum likelihood estimates has been studied by Swaminathan and Gifford (1979). These authors have found that maximum likelihood estimates of θ_i and b_j approach the true values for N as large as 200 and n as large as 100. Since for these values of N and n, Bayesian estimates can be expected to be the same as maximum likelihood estimates, the study was focused on small values of N and n.

The Bayesian estimates and the maximum likelihood estimates were compared with respect to accuracy. The two sets of estimates were compared with respect to: (a) the mean value of the estimates, as compared with the mean value of the true values; (b) the mean squared error difference between the true values and the estimated values; and, (c) the regression of the true value on the estimated value.

It may be argued that since the joint modes of the posterior distribution were taken as estimates of the parameters, the criterion employed to determine the accuracy of the estimates is incompatible with the loss

function employed to arrive at the estimates. This is a valid argument. However, we are primarily interested in comparing the Bayesian estimates with the maximum likelihood estimates. Since, in one sense, the maximum likelihood estimates can be thought of as the modes of the posterior distribution derived under the assumption that the prior information is vague, comparison of two modal estimates using a different criterion other than that involved in deriving the estimates may be justifiable; particularly since this will not provide an "unfair" advantage to one set of estimates.

Comparison of the two estimation procedures in terms of the regression of true values on the estimates needs some explanation. If τ is the true value of the parameter and E, the estimate, then $E(T|E=e) = \beta_0+\beta_1 e$. If $\beta_0=0$ and $\beta_1=1$, then, it can be concluded that the estimates are unbiased, and hence, the departure from the expected values of β_0 and β_1 can be taken as an indicator of bias. It should be pointed out here that the classical notion of bias is not critical in Bayesian analyses. Nevertheless, comparison of the regression lines will provide a further assessment of the accuracy of the two procedures.

The comparison of the maximum likelihood (ML) procedure and the Bayesian procedure for the conditional estimation of ability θ is provided in Table 1. The first column contains the means of the true values of θ, the ML estimates, and the Bayesian estimates. The second column provides an assessment of accuracy in terms of the mean squared deviation between the estimate $\hat{\theta}$ and the true value, θ_t. The correlations between θ_t and $\hat{\theta}$ for each estimation procedure is displayed in column four, while the regression of θ_t on $\hat{\theta}$ is given in column five.

Table 1

Conditional Estimation of θ: Comparison of the Bayesian Estimate and Maximum Likelihood Estimate

		$\bar{\theta}$			$\Sigma(\hat{\theta}-\theta_t)^2/N$		Correlation		Regression			
									ML		Bayes	
Number of Examinees	Number of Items	True	ML	Bayes	ML	Bayes	ML	Bayes	B_o	B_1	B_o	B_1
20	15	-.334	-.321	-.319	.479	.137	.885	.912	.122	1.326	-.035	.853
	25	-.328	-.241	-.222	.175	.083	.951	.950	.171	1.254	.086	.941
	40	.087	.088	.088	.106	.056	.959	.955	-.020	1.246	-.002	1.039
	50	.292	.433	.393	.138	.048	.979	.981	.045	1.327	.069	1.109
50	15	-.000	.130	.115	.440	.117	.915	.928	.130	1.423	.115	.891
	25	-.159	-.218	-.177	.282	.129	.950	.944	-.016	1.271	-.024	.961
	40	-.089	-.167	-.152	.231	.099	.976	.977	-.040	1.428	-.044	1.218
	50	.171	.255	.257	.246	.091	.982	.985	.021	1.361	.057	1.167

An examination of the correlations between the true values and estimates reveals that, in general, the difference between ML and Bayes estimates, is negligible for relatively large values of N and n. However, for small values of N and/or n, the Bayes estimates correlate better with true values than the ML estimates.

The correlation coefficient, by itself, is not a sufficient indicator of the accuracy of estimation. Clear differences between the Bayesian and ML procedures emerge when we examine the other criteria.

In general, the means of the Bayesian estimates, in comparison with the ML estimates, are closer to the means of the true values. This result can be anticipated if we examine the estimating equations [19]. The estimating equations for ML estimates are:

$$\sum_{j=1}^{n} P_{ij} = r_i \qquad (i=1, \ldots, N).$$

The additional term in the Bayesian estimating equations, $(\theta_i - \theta_.)/\sigma^2$ contributes to the regression of the estimates towards the mean, and hence, the Bayesian estimates are closer to the means of the true values. The only exception occurs with N=20 and n=15, 25. At this point, there is no explanation for this anomolous result. Further replications are clearly necessary to establish this point conclusively.

The most dramatic difference between the Bayesian estimates and the ML estimates is with respect to the mean squared deviations of the estimates from the true values. In general, the mean squared deviations are much smaller for the Bayesian estimates than for the ML estimates. The difference is particularly noticable with small N and n. In these cases, the mean squared deviations for the ML estimates is almost four times as

large as that for the Bayesian estimates. This finding can again be explained by the fact prior information is most helpful in these cases. This, together with the regression effect described previously, results in an increase in the accuracy of the estimation procedure.

An examination of the regressions of true values on estimated vlues also provides some interesting results. In general, the intercepts and the slopes of the Bayesian regressions are closer to zero and one respectively, than the ML regressions. The trend for the intercepts is reversed for large n. In these cases, the intercepts for the ML regressions are closer to zero than the intercepts for the Bayesian regressions. This latter result is interpretable, since the maximum likelihood estimates of θ, for large N and n, approach the true values. However, the trend for small n and N is rather surprising since, as a result of regression towards the mean, the Bayesian estimates can be expected to be "biased." The only explanation for this finding is that the ML procedure is severely biased for small n and N, even more so than the Bayesian procedure.

The above findings, for conditional estimation of θ, appear to be valid for the joint estimation of $\underline{\theta}$ and $\underline{b}$ (Tables 2 and 3). In fact, the results for the joint estimation of $\underline{\theta}$ and $\underline{b}$ favor the Bayesian estimates on all counts for both $\underline{\theta}$ and $\underline{b}$: the means of the estimates are closer to the means of the true values; the mean squared deviations are much smaller (in some cases, one-tenth the size of those for ML estimates); the slopes and intercepts are closer to one and zero respectively (the only exception occurs for large N and n, in which case, the intercepts of the ML regression are closer to zero).

Table 2

Joint Estimation of $\underline{\theta}$ and $\underline{b}$: Comparison of the Bayesian and Maximum Likelihood Estimates of Ability

		$\bar{\theta}$			$\Sigma(\theta_i-\bar{\theta})^2/N$		Correlation		Regression			
									ML		Bayes	
Number of Examinees	Number of Items	True	ML	Bayes	ML	Bayes	ML	Bayes	B_o	B_1	B_o	B_1
20	15	-.334	-.237	-.281	.993	.139	.889	.912	.325	1.683	.004	.852
	25	-.224	-.173	-.224	.991	.134	.938	.938	.230	1.802	.007	1.034
	40	.049	.089	.064	.498	.077	.970	.970	.007	1.654	.009	1.115
	50	-.282	-.126	-.104	.453	.144	.969	.971	.373	1.765	.267	1.317
50	15	-.000	.160	.074	.956	.115	.914	.928	.160	1.792	.075	.954
	25	-.117	-.090	-.091	1.015	.209	.925	.925	.111	1.719	.037	1.089
	40	-.089	-.235	-.225	.664	.227	.977	.977	-.074	1.810	-.101	1.409
	50	.171	.302	.280	.776	.238	.984	.985	.006	1.730	.046	1.362

Table 3

Joint Estimation of $\underline{\theta}$ and $\underline{b}$: Comparison of the Bayesian and Maximum Likelihood Estimates of Item Parameters

		$\bar{b}$			$\Sigma(b-\bar{b})^2/N$		Correlation		Regression			
									ML		Bayes	
Number of Examinees	Number of Items	True	ML	Bayes	ML	Bayes	ML	Bayes	B_0	B_1	B_0	B_1
20	15	.050	.279	.083	.880	.096	.946	.939	.187	1.827	.035	.944
	25	-.066	.264	.100	1.513	.153	.929	.954	.399	2.061	.176	1.162
	40	.080	1.008	.014	1.298	.307	.907	.909	-.130	1.723	-.075	1.109
	50	-.127	.005	.018	.794	.190	.933	.933	.207	1.592	.148	1.033
50	15	.050	.132	.033	.775	.099	.983	.985	.035	1.931	-.031	1.281
	25	-.030	.087	.051	1.765	.512	.982	.980	.147	1.978	.096	1.476
	40	.109	.166	.093	.869	.307	.961	.960	-.025	1.741	-.056	1.357
	50	-.049	-.076	-.069	.810	.303	.972	.974	.011	1.783	.001	1.431

DISCUSSION

The Bayesian procedure for estimating parameters in the one-parameter latent trait model is an attractive alternative to the maximum likelihood procedure. Bayesian procedures are conceptually more appealing since direct interpretations of probability statements involving the parameters are possible. Empirically, as the results of the comparison study indicate, the Bayesian estimates of the parameters are superior to the maximum likelihood estimates in terms of their accuracy.

Although the empirical results demonstrate the effectiveness of the Bayesian procedure, it may be argued, and correctly, that the simulation of the data favored the Bayesian procedure. The data were generated to meet the strong distributional assumptions required by the Bayesian procedure. In addition, in specifying prior belief about the distribution ϕ_θ and ϕ_b, s_θ^2 and s_b^2 were set equal to one with the corresponding ν_θ and ν_b being specified as 15. In the simulation ϕ_θ and ϕ_b were set at one, and the specification of s_θ^2, s_b^2 and the relatively large values for ν_θ and ν_b reproduced the true state of affairs. It is not surprising, therefore, that the Bayesian procedure proved to be superior to the maximum likelihood procedure.

In fairness to the study, it should be pointed out that the simulation and the accurate specification of prior belief were deliberate in order to determine the applicability of the Bayesian procedure, at least, under ideal conditions. Preliminary investigations with non-normal data and also with poor specification of priors indicate that the Bayesian procedure, being based on a hierarchical model, is relatively robust and is superior to the maximum likelihood procedure. A detailed study of the effects of poor specification of priors and departures from underlying

assumptions is currently under way and we expect to report these results in the near future.

Despite the encouraging results obtained, a theoretical problem still remains with the estimation procedure. The procedure described in this paper requires the joint estimation of n structural parameters and N incidental parameters. If $N \to \infty$ while n remains fixed, the joint posterior pdf may not become concentrated about the estimated values. This trend is evident from Tables 2 and 3; with increasing N, the intercept and slope do not tend to zero and one respectively. This problem is similar to the one that exists with maximum likelihood estimates. Although from a Bayesian point of view asymptotic properties, such as consistency, are not critical, the lack of them, at least to some degree, is discomforting. It appears that this situation can be remedied, if when estimating the n structural, or item, parameters, the ability parameters are considered nuisance parameters and can be integrated out to yield the marginal posterior pdf of $\underline{b}$. The marginal posterior pdf is currently not available as a result of the exceedingly complex form of the joint posterior pdf. Approximations, such as the one indicated (Equation [15]) may be employed to simplify the joint posterior pdf. Initial investigations reveal that this approximation is reasonably good, but further research in this area is clearly needed.

In summary, we note that the Bayesian procedure developed in this paper is relatively simple to implement, and computationally as efficient as the maximum likelihood procedure. Despite the issues raised above, the Bayesian procedure has the potential for greatly improving the accuracy of the estimates. Moreover, the maximum improvement in accuracy occurs

for small values of N and n, a result that can be expected, and this makes the Bayesian procedure more attractive than the maximum likelihood procedure.

References

Andersen, E. B. Asymptotic properties of conditional maximum likelihood estimates. The Journal of the Royal Statistical Society, Series B, 1970, 32, 283-301.

Andersen, E. B. The numerical solution of a set of conditional equations. The Journal of the Royal Statistical Society, Series B, 1972, 34, 42-54.

Andersen, E. B. Conditional inference in multiple choice questionnaire. British Journal of Mathematical and Statistical Psychology, 1973, 26, 31-44. (a)

Andersen, E. B. A goodness of fit test for the Rasch model. Psychometrika, 1973, 28, 123-140. (b)

Birnbaum, A. Statistical theory for logistic mental test models with a prior distribution of ability. Journal of Mathematical Psychology, 1969, 6, 250-276.

Bock, R. D. Estimating item parameters and latent ability when responses are scored in two or more nominal categories. Psychometrika, 1972, 37, 29-51.

Bock, R. D., & Lieberman, M. Fitting a response model for n dichotomously scored items. Psychometrika, 1970, 35, 179-197.

Hambleton, R. K., & Rovinelli, R. A FORTRAN IV program for generating examinee response data from logistic test models. Behavioral Science, 1973, 18, 74.

Hambleton, R. K., Swaminathan, H., Cook. L., Eignor, D., & Gifford, J. A. Developments in latent trait theory: Models, technical issues, and applications. Review of Educational Research, 1978, 48, 467-510.

Jeffreys, H. Theory of probability (3rd ed.). Oxford: Clarendon, 1961.

Jensema, C. J. A simple technique for estimating latent trait mental test parameters. Educational and Psychological Measurement, 1976, 36, 705-715.

Lindley, D. V. The estimation of many parameters. Foundations of Statistical Inference. (V. P. Godambe and D. A. Sprott, Eds.) Toronto: Holt, Rinehart, and Winston, 1971, 435-455.

Lindley, D. V., & Smith, A. F. Bayesian estimates for the linear model. Journal of the Royal Statistical Society, Series B, 1972, 34, 1-41.

Lord, F. M. An analysis of verbal Scholastic Aptitude Test using Birnbaum's three-parameter logistic model. Educational and Psychological Measurement, 1968, 28, 989-1020.

Lord, F. M. Estimation of latent ability and item parameters when there are omitted responses. Psychometrika, 1974, 39, 247-264.

Lord, F. M., & Novick, M. R. Statistical theories of mental test scores. Reading, MA: Addison-Wesley, 1968.

Meredith, W., & Kearns, J. Empirical Bayes point estimates of latent trait scores without the knowledge of the trait distribution. Psychometrika, 1973, 38, 533-554.

Novick, M. R. Bayesian considerations in educational information systems. Proceedings of the 1970 Invitational Conference on Testing Problems. Princeton, NJ: Educational Testing Service, 1971.

Novick, M. R., & Jackson, P. H. Statistical methods for educational and psychological research. New York: McGraw-Hill, 1974.

Novick, M. R., Lewis, C., & Jackson, P. H. The estimation of proportions in n groups. Psychometrika, 1973, 38, 19-46.

Owen, R. A. Bayesian sequential procedure for quantal response in the context of adaptive mental testing. Journal of the American Statistical Association, 1975, 70, 351-356.

Robbins, H. An empirical Bayes approach to statistics. Proceedings of the Third Berkeley Symposium on Math, Statistics, and Probability 1, 157. University of California Press, 1955.

Sanathanan, L., & Blumenthal, S. The logistic model and estimation of latent structure. Journal of American Statistical Association, 1978, 73, 794-799.

Samejima, F. Estimation of latent ability using a response pattern of graded scores. Psychometric Monograph, 1969, No. 17.

Samejima, F. A general model for free-response data. Psychometric Monograph, 1972, No. 18.

Stein, C. M. Confidence sets for the mean of a multivariate normal distribution. Journal of the Royal Statistical Society, Series B, 1962, 24, 265-296.

Swaminathan, H., & Gifford, J. A. Precision of item parameter estimation. Paper presented at the 1979 Computerized Adaptive Testing Conference. Minneapolis, June 1979.

Tiao, G. C., & Zellner, A. Bayes' Theorem and the use of prior knowledge in regression analysis. Biometrika, 1964, 51, 219-230.

Urry, V. W. Approximations to item parameters of mental test models and their uses. Educational and Psychological Measurement, 1976, 34, 253-269.

Wright, B. D. Solving measurement problems with the Rasch model. Journal of Educational Measurement, 1977, 14, 97-116.

Wright, B. D., & Douglas, G. A. Best procedure for sample-free item analysis. Applied Psychological Measurement, 1977, 1, 281-295.

Wright, B. D., & Panchapakesan, N. A procedure for sample-free item analysis. Educational and Psychological Measurement, 1969, 29, 23-48.

Zellner, A. An introduction to Bayesian inference in econometrics. New York: Wiley, 1971.

DISTRIBUTION LIST

Navy

1 Dr. Jack R. Borsting
Provost & Academic Dean
U.S. Naval Postgraduate School
Monterey, CA 93940

1 Dr. Robert Breaux
Code N-711
NAVTRAEQUIPCEN
Orlando, FL 32813

1 Chief of Naval Education and Training
Liason Office
Air Force Human Resource Laboratory
Flying Training Division
WILLIAMS AFB, AZ 85224

1 COMNAVMILPERSCOM (N-6C)
Dept. of Navy
Washington, DC 20370

1 Deputy Assistant Secretary of the Navy
(Manpower)
Office of the Assistant Secretary of
the Navy (Manpower, Reserve Affairs,
and Logistics)
Washington, DC 20350

1 Dr. Richard Elster
Department of Administrative Sciences
Naval Postgraduate School
Monterey, CA 93940

1 DR. PAT FEDERICO
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

1 Mr. Paul Foley
Navy Personnel R&D Center
San Diego, CA 92152

1 Dr. John Ford
Navy Personnel R&D Center
San Diego, CA 92152

Navy

1 Dr. Patrick R. Harrison
Psychology Course Director
LEADERSHIP & LAW DEPT. (7b)
DIV. OF PROFESSIONAL DEVELOPMMENT
U.S. NAVAL ACADEMY
ANNAPOLIS, MD 21402

1 Dr. Norman J. Kerr
Chief of Naval Technical Training
Naval Air Station Memphis (75)
Millington, TN 38054

1 Dr. William L. Maloy
Principal Civilian Advisor for
Education and Training
Naval Training Command, Code 00A
Pensacola, FL 32508

1 Dr. Kneale Marshall
Scientific Advisor to DCNO(MPT)
OP01T
Washington DC 20370

1 CAPT Richard L. Martin, USN
Prospective Commanding Officer
USS Carl Vinson (CVN-70)
Newport News Shipbuilding and Drydock Co
Newport News, VA 23607

1 Dr. James McBride
Navy Personnel R&D Center
San Diego, CA 92152

1 Dr. George Moeller
Head, Human Factors Dept.
Naval Submarine Medical Research Lab
Groton, CN 06340

1 Library
Naval Health Research Center
P. O. Box 85122
San Diego, CA 92138

1 Naval Medical R&D Command
Code 44
National Naval Medical Center
Bethesda, MD 20014

Navy

1 Ted M. I. Yellen
Technical Information Office, Code 201
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

1 Library, Code P201L
Navy Personnel R&D Center
San Diego, CA 92152

1 Technical Director
Navy Personnel R&D Center
San Diego, CA 92152

6 Commanding Officer
Naval Research Laboratory
Code 2627
Washington, DC 20390

1 Psychologist
ONR Branch Office
Bldg 114, Section D
666 Summer Street
Boston, MA 02210

1 Psychologist
ONR Branch Office
536 S. Clark Street
Chicago, IL 60605

1 Office of Naval Research
Code 437
800 N. Quincy SStreet
Arlington, VA 22217

5 Personnel & Training Research Programs
(Code 458)
Office of Naval Research
Arlington, VA 22217

1 Psychologist
ONR Branch Office
1030 East Green Street
Pasadena, CA 91101

1 Special Asst. for Education and
Training (OP-01E)
Rm. 2705 Arlington Annex
Washington, DC 20370

Navy

1 Office of the Chief of Naval Operations
Research, Development, and Studies Branc
(OP-102)
Washington, DC 20350

1 Head, Manpower Training and Reserves
Section (Op-964D)
Room 4A478, The Pentagon
Washington, DC 20350

1 Captain Donald F. Parker, USN
Commanding Officer
Navy Personnel R&D Center
San Diego, CA 92152

1 LT Frank C. Petho, MSC, USN (Ph.D)
Code L51
Naval Aerospace Medical Research Laborat
Pensacola, FL 32508

1 The Principal Deputy Assistant
Secretary of the Navy (MRA&L)
4E780, The Pentagon
Washington, DC 20350

1 Director, Research & Analysis Division
Plans and Policy Department
Navy Recruiting Command
4015 Wilson Boulevard
Arlington, VA 22203

1 Dr. Bernard Rimland (03B)
Navy Personnel R&D Center
San Diego, CA 92152

1 Mr. Arnold Rubenstein
Naval Personnel Support Technology
Naval Material Command (08T244)
Room 1044, Crystal Plaza #5
2221 Jefferson Davis Highway
Arlington, VA 20360

1 Dr. Worth Scanland
Chief of Naval Education and Training
Code N-5
NAS, Pensacola, FL 32508

Navy

1 Dr. Robert G. Smith
Office of Chief of Naval Operations
OP-987H
Washington, DC 20350

1 Dr. Alfred F. Smode
Training Analysis & Evaluation Group (TAEG)
Dept. of the Navy
Orlando, FL 32813

1 Dr. Richard Sorensen
Navy Personnel R&D Center
San Diego, CA 92152

1 W. Gary Thomson
Naval Ocean Systems Center
Code 7132
San Diego, CA 92152

1 Dr. Ronald Weitzman
Code 54 WZ
Department of Administrative Sciences
U. S. Naval Postgraduate School
Monterey, CA 93940

1 DR. MARTIN F. WISKOFF
NAVY PERSONNEL R& D CENTER
SAN DIEGO, CA 92152

1 Mr John H. Wolfe
Code P310
U. S. Navy Personnel Research and Development Center
San Diego, CA 92152

Army

1 Technical Director
U. S. Army Research Institute for the Behavioral and Social Sciences
5001 Eisenhower Avenue
Alexandria, VA 22333

1 HQ USAREUE & 7th Army
ODCSOPS
USAAREUE Director of GED
APO New York 09403

1 DR. RALPH DUSEK
U.S. ARMY RESEARCH INSTITUTE
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

1 Dr. Myron Fischl
U.S. Army Research Institute for the Social and Behavioral Sciences
5001 Eisenhower Avenue
Alexandria, VA 22333

1 Dr. Michael Kaplan
U.S. ARMY RESEARCH INSTITUTE
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

1 Dr. Milton S. Katz
Training Technical Area
U.S. Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

1 Dr. Harold F. O'Neil, Jr.
Attn: PERI-OK
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

1 DR. JAMES L. RANEY
U.S. ARMY RESEARCH INSTITUTE
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

1 Mr. Robert Ross
U.S. Army Research Institute for the Social and Behavioral Sciences
5001 Eisenhower Avenue
Alexandria, VA 22333

Army

1 Dr. Robert Sasmor
U. S. Army Research Institute for the Behavioral and Social Sciences
5001 Eisenhower Avenue
Alexandria, VA 22333

1 Commandant
US Army Institute of Administration
Attn: Dr. Sherrill
FT Benjamin Harrison, IN 46256

1 Dr. Frederick Steinheiser
U. S. Army Reserch Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

1 Dr. Joseph Ward
U.S. Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Air Force

1 Air Force Human Resources Lab
AFHRL/MPD
Brooks AFB, TX 78235

1 U.S. Air Force Office of Scientific Research
Life Sciences Directorate, NL
Bolling Air Force Base
Washington, DC 20332

1 Air University Library
AUL/LSE 76/443
Maxwell AFB, AL 36112

1 Dr. Earl A. Alluisi
HQ, AFHRL (AFSC)
Brooks AFB, TX 78235

1 Dr. Genevieve Haddad
Program Manager
Life Sciences Directorate
AFOSR
Bolling AFB, DC 20332

1 Dr. Ross L. Morgan (AFHRL/LR)
Wright -Patterson AFB
Ohio 45433

1 Research and Measurment Division
Research Branch, AFMPC/MPCYPR
Randolph AFB, TX 78148

1 Dr. Malcolm Ree
AFHRL/MP
Brooks AFB, TX 78235

1 Dr. Marty Rockway (AFHRL/TT)
Lowry AFB
Colorado 80230

1 Dr. Frank Schufletowski
U.S. Air Force
ATC/XPTD
Randolph AFB, TX 78148

1 Jack A. Thorpe, Maj., USAF
Naval War College
Providence, RI 02846

Air Force

1 Dr. Joe Ward, Jr.
AFHRL/MPMD
Brooks AFB, TX 78235

Marines

1 H. William Greenup
Education Advisor (E031)
Education Center, MCDEC
Quantico, VA 22134

1 Director, Office of Manpower Utilization
HQ, Marine Corps (MPU)
BCB, Bldg. 2009
Quantico, VA 22134

1 Headquarters, U. S. Marine Corps
Code MPI-20
Washington, DC 20380

1 Major Michael L. Patrow, USMC
Headquarters, Marine Corps
(Code MPI-20)
Washington, DC 20380

1 DR. A.L. SLAFKOSKY
SCIENTIFIC ADVISOR (CODE RD-1)
HQ, U.S. MARINE CORPS
WASHINGTON, DC 20380

CoastGuard

1 Chief, Psychological Reserch Branch
U. S. Coast Guard (G-P-1/2/TP42)
Washington, DC 20593

1 Mr. Thomas A. Warm
U. S. Coast Guard Institute
P. O. Substation 18
Oklahoma City, OK 73169

Other DoD

12 Defense Documentation Center
Cameron Station, Bldg. 5
Alexandria, VA 22314
Attn: TC

1 Dr. Dexter Fletcher
ADVANCED RESEARCH PROJECTS AGENCY
1400 WILSON BLVD.
ARLINGTON, VA 22209

1 Dr. William Graham
Testing Directorate
MEPCOM/MEPCT-P
Ft. Sheridan, IL 60037

1 Director, Research and Data
OASD(MRA&L)
3B919, The Pentagon
Washington, DC 20301

1 Military Assistant for Training and
Personnel Technology
Office of the Under Secretary of Defense
for Research & Engineering
Room 3D129, The Pentagon
Washington, DC 20301

1 MAJOR Wayne Sellman, USAF
Office of the Assistant Secretary
of Defense (MRA&L)
3B930 The Pentagon
Washington, DC 20301

Civil Govt

1 Dr. Susan Chipman
Learning and Development
National Institute of Education
1200 19th Street NW
Washington, DC 20208

1 Dr. Lorraine D. Eyde
Personnel R&D Center
Office of Personnel Management of USA
1900 EStreet NW
Washington, D.C. 20415

1 Jerry Lehnus
REGIONAL PSYCHOLOGIST
U.S. Office of Personnel Management
230 S. DEARBORN STREET
CHICAGO, IL 60604

1 Dr. Andrew R. Molnar
Science Education Dev.
and Research
National Science Foundation
Washington, DC 20550

1 Personnel R&D Center
Office of Personnel Managment
1900 E Street NW
Washington, DC 20415

1 Dr. H. Wallace Sinaiko
Program Director
Manpower Research and Advisory Services
Smithsonian Institution
801 North Pitt Street
Alexandria, VA 22314

1 Dr. Vern W. Urry
Personnel R&D Center
Office of Personnel Management
1900 E Street NW
Washington, DC 20415

1 Dr. Joseph L. Young, Director
Memory & Cognitive Processes
National Science Foundation
Washington, DC 20550

Non Govt

1 Dr. Erling B. Andersen
Department of Statistics
Studiestraede 6
1455 Copenhagen
DENMARK

1 1 psychological research unit
Dept. of Defense (Army Office)
Campbell Park Offices
Canberra ACT 2600, Australia

1 Dr. Alan Baddeley
Medical Research Council
Applied Psychology Unit
15 Chaucer Road
Cambridge CB2 2EF
ENGLAND

1 Dr. Isaac Bejar
Educational Testing Service
Princeton, NJ 08450

1 Dr. Werner Birke
DezWPs im Streitkraefteamt
Postfach 20 50 03
D-5300 Bonn 2
WEST GERMANY

1 Dr. R. Darrel Bock
Department of Education
University of Chicago
Chicago, IL 60637

1 Dr. Nicholas A. Bond
Dept. of Psychology
Sacramento State College
600 Jay Street
Sacramento, CA 95819

1 Dr. Robert Brennan
American College Testing Programs
P. O. Box 168
Iowa City, IA 52240

1 DR. C. VICTOR BUNDERSON
WICAT INC.
UNIVERSITY PLAZA, SUITE 10
1160 SO. STATE ST.
OREM, UT 84057

Non Govt

1 Dr. John B. Carroll
Psychometric Lab
Univ. of No. Carolina
Davie Hall 013A
Chapel Hill, NC 27514

1 Charles Myers Library
Livingstone House
Livingstone Road
Stratford
London E15 2LJ
ENGLAND

1 Dr. Kenneth E. Clark
College of Arts & Sciences
University of Rochester
River Campus Station
Rochester, NY 14627

1 Dr. Norman Cliff
Dept. of Psychology
Univ. of So. California
University Park
Los Angeles, CA 90007

1 Dr. William E. Coffman
Director, Iowa Testing Programs
334 Lindquist Center
University of Iowa
Iowa City, IA 52242

1 Dr. Meredith P. Crawford
American Psychological Association
1200 17th Street, N.W.
Washington, DC 20036

1 Dr. Hans Crombag
Education Research Center
University of Leyden
Boerhaavelaan 2
2334 EN Leyden
The NETHERLANDS

1 LCOL J. C. Eggenberger
DIRECTORATE OF PERSONNEL APPLIED RESEARC
NATIONAL DEFENCE HQ
101 COLONEL BY DRIVE
OTTAWA, CANADA K1A OK2

Non Govt

1 Dr. A. J. Eschenbrenner
Dept. E422, Bldg. 81
McDonnell Douglas Astronautics Co.
P.O.Box 516
St. Louis, MO 63166

1 Dr. Leonard Feldt
Lindquist Center for Measurment
University of Iowa
Iowa City, IA 52242

1 Dr. Richard L. Ferguson
The American College Testing Program
P.O. Box 168
Iowa City, IA 52240

1 Dr. Victor Fields
Dept. of Psychology
Montgomery College
Rockville, MD 20850

1 Univ. Prof. Dr. Gerhard Fischer
Liebiggasse 5/3
A 1010 Vienna
AUSTRIA

1 Professor Donald Fitzgerald
University of New England
Armidale, New South Wales 2351
AUSTRALIA

1 Dr. Edwin A. Fleishman
Advanced Research Resources Organ.
Suite 900
4330 East West Highway
Washington, DC 20014

1 Dr. John R. Frederiksen
Bolt Beranek & Newman
50 Moulton Street
Cambridge, MA 02138

1 DR. ROBERT GLASER
LRDC
UNIVERSITY OF PITTSBURGH
3939 O'HARA STREET
PITTSBURGH, PA 15213

Non Govt

1 Dr. Daniel Gopher
Industrial & Management Engineering
Technion-Israel Institute of Technology
Haifa
ISRAEL

1 Dr. Ron Hambleton
School of Education
University of Massechusetts
Amherst, MA 01002

1 Dr. Chester Harris
School of Education
University of California
Santa Barbara, CA 93106

1 Glenda Greenwald, Ed.
"Human Intelligence Newsletter"
P. O. Box 1163
Birmingham, MI 48012

1 Dr. Lloyd Humphreys
Department of Psychology
University of Illinois
Champaign, IL 61820

1 Library
HumRRO/Western Division
27857 Berwick Drive
Carmel, CA 93921

1 Dr. Steven Hunka
Department of Education
University of Alberta
Edmonton, Alberta
CANADA

1 Dr. Earl Hunt
Dept. of Psychology
University of Washington
Seattle, WA 98105

1 Dr. Huynh Huynh
College of Education
University of South Carolina
Columbia, SC 29208

Non Govt

1 Dr. Douglas H. Jones
Rm T-255
Educational Testing Service
Princeton, NJ 08450

3 Journal Supplement Abstract Service
American Psychological Association
1200 17th Street N.W.
Washington, DC 20036

1 Professor John A. Keats
University of Newcastle
AUSTRALIA 2308

1 Dr. Stephen Kosslyn
Harvard University
Department of Psychology
33 Kirkland Street
Cambridge, MA 02138

1 Mr. Marlin Kroger
1117 Via Goleta
Palos Verdes Estates, CA 90274

1 Dr. Alan Lesgold
Learning R&D Center
University of Pittsburgh
Pittsburgh, PA 15260

1 Dr. Michael Levine
210 Education Building
University of Illinois
Champaign, IL 61820

1 Dr. Charles Lewis
Faculteit Sociale Wetenschappen
Rijksuniversiteit Groningen
Oude Boteringestraat
Groningen
NETHERLANDS

1 Dr. Robert Linn
College of Education
University of Illinois
Urbana, IL 61801

1 Dr. Frederick M. Lord
Educational Testing Service
Princeton, NJ 08540

Non Govt

1 Dr. James Lumsden
Department of Psychology
University of Western Australia
Nedlands W.A. 6009
AUSTRALIA

1 Dr. Gary Marco
Educational Testing Service
Princeton, NJ 08450

1 Dr. Scott Maxwell
Department of Psychology
University of Houston
Houston, TX 77004

1 Dr. Samuel T. Mayo
Loyola University of Chicago
820 North Michigan Avenue
Chicago, IL 60611

1 Dr. Mark Miller
Computer Science Laboratory
Texas Instruments, Inc.
Mail Station 371, P.O. Box 225936
Dallas, TX 75265

1 Professor Jason Millman
Department of Education
Stone Hall
Cornell University
Ithaca, NY 14853

1 Dr. Allen Munro
Behavioral Technology Laboratories
1845 Elena Ave., Fourth Floor
Redondo Beach, CA 90277

1 Dr. Melvin R. Novick
356 Lindquist Center for Measurment
University of Iowa
Iowa City, IA 52242

1 Dr. Jesse Orlansky
Institute for Defense Analyses
400 Army Navy Drive
Arlington, VA 22202

Non Govt

1 Dr. James A. Paulson
Portland State University
P.O. Box 751
Portland, OR 97207

1 MR. LUIGI PETRULLO
2431 N. EDGEWOOD STREET
ARLINGTON, VA 22207

1 DR. DIANE M. RAMSEY-KLEE
R-K RESEARCH & SYSTEM DESIGN
3947 RIDGEMONT DRIVE
MALIBU, CA 90265

1 MINRAT M. L. RAUCH
P II 4
BUNDESMINISTERIUM DER VERTEIDIGUNG
POSTFACH 1328
D-53 BONN 1, GERMANY

1 Dr. Mark D. Reckase
Educational Psychology Dept.
University of Missouri-Columbia
4 Hill Hall
Columbia, MO 65211

1 Dr. Andrew M. Rose
American Institutes for Research
1055 Thomas Jefferson St. NW
Washington, DC 20007

1 Dr. Leonard L. Rosenbaum, Chairman
Department of Psychology
Montgomery College
Rockville, MD 20850

1 Dr. Lawrence Rudner
403 Elm Avenue
Takoma Park, MD 20012

1 Dr. J. Ryan
Department of Education
University of South Carolina
Columbia, SC 29208

1 PROF. FUMIKO SAMEJIMA
DEPT. OF PSYCHOLOGY
UNIVERSITY OF TENNESSEE
KNOXVILLE, TN 37916

Non Govt

1 DR. ROBERT J. SEIDEL
INSTRUCTIONAL TECHNOLOGY GROUP
HUMRRO
300 N. WASHINGTON ST.
ALEXANDRIA, VA 22314

1 Dr. Kazuo Shigemasu
University of Tohoku
Department of Educational Psychology
Kawauchi, Sendai 980
JAPAN

1 Dr. Edwin Shirkey
Department of Psychology
University of Central Florida
Orlando, FL 32816

1 Dr. Richard Snow
School of Education
Stanford University
Stanford, CA 94305

1 Dr. Kathryn T. Spoehr
Department of Psychology
Brown University
Providence, RI 02912

1 Dr. Robert Sternberg
Dept. of Psychology
Yale University
Box 11A, Yale Station
New Haven, CT 06520

1 DR. PATRICK SUPPES
INSTITUTE FOR MATHEMATICAL STUDIES IN
THE SOCIAL SCIENCES
STANFORD UNIVERSITY
STANFORD, CA 94305

1 Dr. Hariharan Swaminathan
Laboratory of Psychometric and
Evaluation Research
School of Education
University of Massachusetts
Amherst, MA 01003

Non Govt

1 Dr. Brad Sympson
Psychometric Research Group
Educational Testing Service
Princeton, NJ 08541

1 Dr. Kikumi Tatsuoka
Computer Based Education Research
Laboratory
252 Engineering Research Laboratory
University of Illinois
Urbana, IL 61801

1 Dr. David Thissen
Department of Psychology
University of Kansas
Lawrence, KS 66044

1 Dr. J. Uhlaner
Perceptronics, Inc.
6271 Variel Avenue
Woodland Hills, CA 91364

1 Dr. Howard Wainer
Bureau of Social SCience Research
1990 M Street, N. W.
Washington, DC 20036

1 Dr. David J. Weiss
N660 Elliott Hall
University of Minnesota
75 E. River Road
Minneapolis, MN 55455

1 DR. GERSHON WELTMAN
PERCEPTRONICS INC.
6271 VARIEL AVE.
WOODLAND HILLS, CA 91367

1 DR. SUSAN E. WHITELY
PSYCHOLOGY DEPARTMENT
UNIVERSITY OF KANSAS
LAWRENCE, KANSAS 66044

1 Wolfgang Wildgrube
Streitkraefteamt
Box 20 50 03
D-5300 Bonn 2
WEST GERMANY

Non Govt

1 Dr. J. Arthur Woodward
Department of Psychology
University of California

ILMED

-8